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Waves on the beta-plane over sparse topography

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This paper deals with linear waves on the beta-plane over topography. The main assumption is that the topography consists of isolated radially symmetric irregularities (random or periodic), such that their characteristic radii are much smaller than the distances between them. This approximation allows one to obtain the dispersion relation for the frequency of wave modes; and in order to examine the properties of those, we consider a particular case where bottom irregularities are cylinders of various heights and radii. It is demonstrated that if all irregularities are of the same height, h, there exist two topographic and one Rossby modes. The frequency of one of the topographic modes is 'locked' inside the band $(-fh/2H_0, fh/2H_0)$, where f is the Coriolis parameter and H_0 is the mean depth of the ocean. The frequencies of the other topographic mode and the barotropic Rossby mode are 'locked' above and below the band, respectively. It is also demonstrated that if the heights of cylinders are distributed within a certain range, $(-h_0, h_0)$, no harmonic modes exist with frequencies inside the interval $(-fh_0/2H_0, fh_0/2H_0)$. The topographic and Rossby modes are 'pushed' out of the 'prohibited' band.

1. Introduction

Bottom irregularities exist everywhere in the ocean and play an important role in the dynamics of oceanic waves. Even small-amplitude, short-horizontal-scale topography can strongly affect planetary motions and support the so-called topographic wave modes. The latter type of wave was discovered by Rhines & Bretherton (1973) for the case of one-dimensional topography (i.e. such that the isobaths are straight lines). Since then, their results have been generalized for the two-layer case (McWilliams 1974; Samelson 1992; Reznik & Tsybaneva 2000), the linearly stratified case (Bobrovich & Reznik 2000), and for waves in zonal currents (Benilov 2000*a*, *b*).

It turned out, however, that the natural extension of Rhines & Bretherton's (1973) results to the general case of two-dimensional topography is much more difficult. There are only two papers examining this case. In their original work, Rhines & Bretherton (1973) considered the 'quasi-one-dimensional' case, where one of the horizontal scales of topography is much larger than the other. A particular case of fully two-dimensional topography was examined numerically by Samelson (1992) – however, this author concentrated on bottom irregularities with horizontal scales comparable to the wavelength, i.e. it was not short-scale topography which is the subject of this work.

The present paper examines two-dimensional short-scale topography. The main assumption, which has enabled us to make a notable advance, is that the topography is sparse, i.e. consists of isolated, radially symmetric features separated by large distances. In § 2, we introduce and scale the governing equations. In § 3 we perform the usual (similar to Rhines & Bretherton 1973) asymptotic analysis and demonstrate why the two-dimensional case is much more difficult than the one-dimensional case. In § 4, the notion of sparse topography is introduced and used to derive the dispersion relation for the topographic and barotropic Rossby waves. Finally, in § 5, we present a detailed study of the particular case where the bottom irregularities are cylinders of various heights and radii.

2. Governing equations and scaling

The standard linear equation which describes quasi-geostrophic (QG) flows on the beta-plane over topography is

$$\nabla^2\psi_t + \frac{f}{H_0}J(\psi, d) + \beta\psi_x = 0, \tag{2.1}$$

where (x, y) and t are the Cartesian coordinates and time, $\psi(x, y, t)$ is the streamfunction, f and β are the Coriolis parameter and its meridional gradient, and $d(x, y)$ is the deviation of the ocean depth $H(x, y)$ from its mean value H_0:

$$d = H - H_0.$$

We are concerned with oscillations that are harmonic in t, but not in x and y. However, for convenience, we shall separate the harmonic dependence on the spatial variables from the dependence 'induced' by topography:

$$\psi(x, y, t) = \phi(x, y)\exp(\mathrm{i}kx + \mathrm{i}ly - \mathrm{i}\omega t), \tag{2.2}$$

where ω and (k, l) are the frequency and wavevector, and $\phi(x, y)$ describes the effect of bottom irregularities (in particular, if $d = 0$, then $\phi = \text{const}$). Substituting (2.2) into (2.1), we obtain

$$-\mathrm{i}\omega[\nabla^2\phi + 2i(k\phi_x + l\phi_y) - (k^2 + l^2)\phi] + \frac{f}{H_0}[\phi_x d_y - \phi_y d_x + \mathrm{i}(kd_y - ld_x)\phi] + \beta(\phi_x + \mathrm{i}k\phi) = 0. \tag{2.3}$$

Next, we introduce non-dimensional variables:

$$x_* = x/L, \quad y_* = y/L, \quad k_* = k\Lambda \quad l_* = l\Lambda, \quad \omega_* = \omega T, \quad d_* = d/D,$$

where L and D are the horizontal spatial scale and characteristic height of the topography, and Λ and T are the wavelength and period of the wave mode. Rewriting (2.3) in terms of the new variables and omitting asterisk, we obtain

$$\begin{aligned}\varepsilon_\omega\omega[\nabla^2\phi + 2\mathrm{i}\varepsilon_k(k\phi_x + l\phi_y) - \varepsilon_k^2(k^2 + l^2)\phi] \\ +\varepsilon_d[\mathrm{i}(\phi_x d_y - \phi_y d_x) - \varepsilon_k(kd_y - ld_x)\phi] + \varepsilon_\beta(\mathrm{i}\phi_x - \varepsilon_k k\phi) = 0,\end{aligned} \tag{2.4}$$

where

$$\varepsilon_\omega = \frac{1}{Tf}, \quad \varepsilon_k = \frac{L}{\Lambda}, \quad \varepsilon_d = \frac{D}{H_0}, \quad \varepsilon_\beta = \frac{\beta L}{f}. \tag{2.5}$$

Following the usual approximation of a 'rough-bottomed' ocean (Rhines & Bretherton 1973), we introduce the 'formal' small parameter ε and assume that

$$\varepsilon_\omega = \varepsilon_k = \varepsilon_d = \varepsilon, \quad \varepsilon_\beta = \alpha\varepsilon^2, \tag{2.6}$$

where $\alpha = O(1)$ is a constant. (Assumption (2.6) corresponds to small-amplitude, short-scale bottom irregularities with parameters $L \sim 5 - 10\,\text{km}$, $D = 100 - 300\,\text{m}$ – see Rhines & Bretherton 1973.)

Substituting (2.6) into (2.4), we obtain

$$\omega\nabla^2\phi + \mathrm{i}(d_y\phi_x - d_x\phi_y) = \varepsilon[(kd_y - ld_x)\phi - 2\mathrm{i}\omega(k\phi_x + l\phi_y) - \mathrm{i}\alpha\phi_x] + \varepsilon^2[\alpha k + \omega(k^2 + l^2)]\phi. \quad (2.7)$$

3. Asymptotic analysis

Let us expand the eigenfunction and eigenvalue of equation (2.7) in powers of ε:

$$\phi = \phi^{(0)} + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(2)} + \dots, \qquad \omega = \omega^{(0)} + \varepsilon\omega^{(1)} + \varepsilon^2\omega^{(2)} + \dots .$$

At the zeroth order, we can assume that the wave mode is not affected by topography:

$$\phi^{(0)} = 1. \quad (3.1)$$

At the next order, we obtain the following equation for $\phi^{(1)}$:

$$\omega^{(0)}\nabla^2\phi^{(1)} + \mathrm{i}(d_y\phi_x^{(1)} - d_x\phi_y^{(1)}) = kd_y - ld_x. \quad (3.2)$$

If the isobaths are straight lines, i.e. if

$$d = d(\xi), \qquad \xi = \mathrm{const}_1\, x + \mathrm{const}_2\, y,$$

the solution to equation (3.2) can be sought in the form

$$\phi^{(1)}(x, y) = \phi^{(1)}(\xi).$$

The Jacobian in (3.2) disappears, and the resulting equation can be readily solved (see Rhines & Bretherton 1973).

In the general case, where $d(x, y)$ is an arbitrary function, (3.2) has no 'simple' solutions and thus becomes the main stumbling block of the two-dimensional problem. We shall postpone its discussion until the next section, and proceed to the next order of perturbation expansion. Taking into account (3.1), we obtain

$$\omega^{(0)}\nabla^2\phi^{(2)} + \omega^{(1)}\nabla^2\phi^{(1)} + \mathrm{i}(d_y\phi_x^{(2)} - d_x\phi_y^{(2)}) = (kd_y - ld_x)\phi^{(1)} - 2\mathrm{i}\omega^{(0)}(k\phi_x^{(1)} + l\phi_y^{(1)}) - \mathrm{i}\alpha\phi_x^{(1)} + \alpha k + \omega^{(0)}(k^2 + l^2). \quad (3.3)$$

As usually happens, the solution $\phi^{(2)}$ of equation (3.3) is not necessarily bounded at infinity. To demonstrate this, average (3.3) over the (x, y)-plane, i.e. consider

$$\lim_{S\to\infty} \frac{1}{4S^2} \int_{-S}^{S}\int_{-S}^{S} (3.3)\ \mathrm{d}x\,\mathrm{d}y.$$

Integrating by parts and assuming that $\phi^{(1)}$ and $\phi^{(2)}$ are bounded as $x, y \to \infty$, one can see that the left-hand side of (3.3) vanishes when $S \to \infty$, and we end up with

$$\langle (kd_y - ld_x)\phi^{(1)} - 2\mathrm{i}\omega^{(0)}(k\phi_x^{(1)} + l\phi_y^{(1)}) - \mathrm{i}\alpha\phi_x^{(1)} + \alpha k + \omega^{(0)}(k^2 + l^2)\rangle = 0,$$

where $\langle\dots\rangle$ denotes spatial averaging. The second and third terms in this equation are full derivatives and therefore have zero average, which yields the following equality:

$$\langle (kd_y - ld_x)\phi^{(1)}\rangle + \alpha k + \omega^{(0)}(k^2 + l^2) = 0. \quad (3.4)$$

Dispersion relation (3.4) and the 'definition' of $\phi^{(1)}$ (equation (3.2)), determine the frequency of the wave modes (if any).

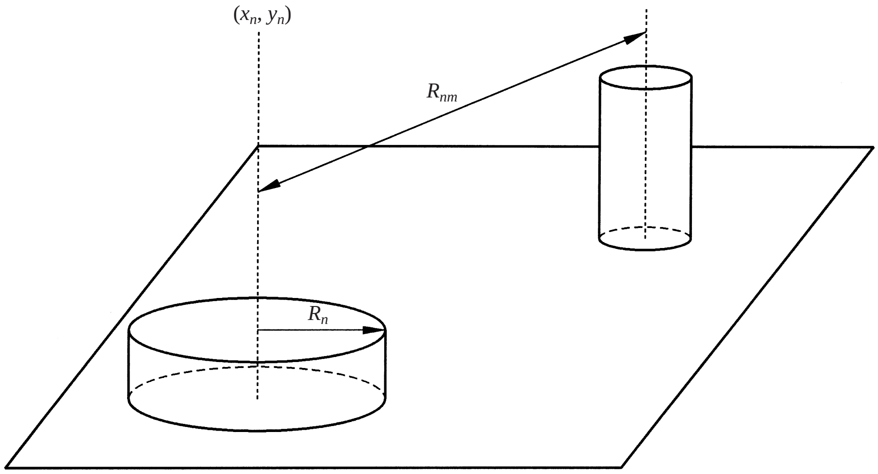

FIGURE 1. The formulation of the problem: an oceanic bottom with radially symmetric (but not necessarily cylindrical) isolated irregularities.

4. The approximation of sparse topography

Equation (3.2) is a linear PDE with variable coefficients and it does not appear to have an analytical solution in the general case (at least, no-one has been able to obtain one since 1973, when Rhines & Bretherton derived (3.2)). In the present paper, (3.2) will be analysed asymptotically for the case of sparse topography.

Assume that the topography consists of an infinite number of radially symmetric irregularities

$$d(x, y) = \sum_{n=1}^{\infty} f_n(r_n), \tag{4.1}$$

where

$$r_n = \sqrt{(x - x_n)^2 + (y - y_n)^2}$$

and (x_n, y_n) are the coordinates of the centre of the nth irregularity (see figure 1). We shall also assume for simplicity that the irregularities are localized within circles of various radii:

$$f_n = 0 \quad \text{if } r_n > R_n.$$

Introducing the distance between the centres of irregularities,

$$R_{nm} = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2},$$

we assume that they are rare and far apart, i.e.

$$\overline{R}_n \ll \overline{R}_{nm},$$

where $\overline{R}_n$ and $\overline{R}_{nm}$ are the typical values of R_n and R_{nm} respectively.† Thus, the solution to (3.2) can be approximated by a sum of the contributions of individual

† It should be emphasized that both $\overline{R}_{nm}$ and $\overline{R}_n$ should be smaller than the wavelength (which was our original assumption), i.e. $\overline{R}_n \ll \overline{R}_{nm} \ll \Lambda$.

irregularities:

$$\phi^{(1)} \approx \sum_{n=1}^{\infty} \phi_n^{(1)}(r_n), \tag{4.2}$$

$$\omega^{(0)}\nabla^2\phi_n^{(1)} + \mathrm{i}\left(\frac{\partial f_n}{\partial y}\frac{\partial \phi_n^{(1)}}{\partial x} - \frac{\partial f_n}{\partial x}\frac{\partial \phi_n^{(1)}}{\partial y}\right) = k\frac{\partial f_n}{\partial y} - l\frac{\partial f_n}{\partial x}. \tag{4.3}$$

Rewriting (4.3) in terms of the polar variables associated with the centre of the nth irregularity,

$$x = x_n + r_n \cos\theta_n, \qquad y = y_n + r_n \sin\theta_n,$$

we obtain

$$\omega^{(0)}\left[\frac{1}{r_n}\frac{\partial}{\partial r_n}\left(r_n\frac{\partial \phi_n^{(1)}}{\partial r_n}\right) + \frac{1}{r_n}\frac{\partial^2 \phi_n^{(1)}}{\partial \theta_n^2}\right] - \mathrm{i}\frac{1}{r_n}\frac{df_n}{dr_n}\frac{\partial \phi_n^{(1)}}{\partial \theta_n} = (k\sin\theta_n - l\cos\theta_n)\frac{\mathrm{d}f_n}{\mathrm{d}r_n}.$$

It is clear that $\phi_n^{(1)}$ can be represented in the form

$$\phi_n^{(1)}(r_n, \theta_n) = B_n(r_n)\sin\theta_n + C_n(r_n)\cos\theta_n, \tag{4.4}$$

where B_n and C_n satisfy

$$\omega^{(0)}\left[\frac{1}{r_n}\frac{\mathrm{d}}{\mathrm{d}r_n}\left(r_n\frac{\mathrm{d}B_n}{\mathrm{d}r_n}\right) - \frac{1}{r_n^2}B_n\right] + \frac{\mathrm{i}}{r_n}\frac{\mathrm{d}f_n}{\mathrm{d}r_n}C_n = k\frac{\mathrm{d}f_n}{\mathrm{d}r_n}, \tag{4.5}$$

$$\omega^{(0)}\left[\frac{1}{r_n}\frac{\mathrm{d}}{\mathrm{d}r_n}\left(r_n\frac{\mathrm{d}C_n}{\mathrm{d}r_n}\right) - \frac{1}{r_n^2}C_n\right] - \frac{\mathrm{i}}{r_n}\frac{\mathrm{d}f_n}{\mathrm{d}r_n}B_n = -l\frac{\mathrm{d}f_n}{\mathrm{d}r_n}. \tag{4.6}$$

The boundary conditions reflect the usual requirements of regularity at the origin and decay at infinity:

$$B_n(0) = C_n(0) = 0, \tag{4.7}$$

$$B_n, C_n \to 0 \quad \text{as } r_n \to \infty. \tag{4.8}$$

It is also convenient to rewrite the dispersion relation (3.4) in terms of B_n and C_n. Substituting (4.1)–(4.2) and (4.4) into (3.4), we obtain

$$\left\langle \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(k\sin\theta_n - l\cos\theta_n)\frac{\mathrm{d}f_n}{\mathrm{d}r}(B_m\sin\theta_m + C_m\cos\theta_m)\right\rangle + \alpha k + \omega^{(0)}(k^2+l^2) = 0.$$

As the irregularities (described by f_n) are separated by large distances, we can safely omit those terms in this equation for which $m \neq n$:

$$\left\langle \sum_{n=1}^{\infty}(k\sin\theta_n - l\cos\theta_n)\frac{\mathrm{d}f_n}{\mathrm{d}r}(B_n\sin\theta_n + C_n\cos\theta_n)\right\rangle + \alpha k + \omega^{(0)}(k^2+l^2) = 0.$$

Finally, we assume that the spatial average in this equation can be replaced by the mean contribution from a single irregularity, multiplied by the density ρ of irregularities per unit area:

$$\rho\,\overline{\int_0^{\infty}\int_0^{2\pi}(k\sin\theta_n - l\cos\theta_n)\frac{\mathrm{d}f_n}{\mathrm{d}r_n}(B_n\sin\theta_n + C_n\cos\theta_n)r_n\mathrm{d}\theta_n\mathrm{d}r_n} + \alpha k + \omega^{(0)}\left(k^2+l^2\right) = 0,$$

where the bar denotes averaging over the ensemble of f_n. Performing the integration with respect to θ_n, we obtain (subscripts and superscripts omitted)

$$\pi\rho\overline{\int_0^\infty r\frac{\mathrm{d}f}{\mathrm{d}r}(kB-lC)\,\mathrm{d}r}+\alpha k+\omega(k^2+l^2)=0. \tag{4.9}$$

This dispersion relation should be solved with the boundary-value problem (4.5)–(4.8), which we also rewrite without subscripts and superscripts:

$$\omega\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}B}{\mathrm{d}r}\right)-\frac{1}{r^2}B\right]+\frac{\mathrm{i}}{r}\frac{\mathrm{d}f}{\mathrm{d}r}C=k\frac{\mathrm{d}f}{\mathrm{d}r}, \tag{4.10}$$

$$\omega\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}C}{\mathrm{d}r}\right)-\frac{1}{r^2}C\right]-\frac{\mathrm{i}}{r}\frac{\mathrm{d}f}{\mathrm{d}r}B=-l\frac{\mathrm{d}f}{\mathrm{d}r}. \tag{4.11}$$

$$B(0)=C(0)=0, \tag{4.12}$$

$$B,C\to 0\quad \text{as } r\to\infty. \tag{4.13}$$

In the next section, set (4.9)–(4.13) will be solved for a particular case.

5. Cylindrical bottom irregularities

The simplest particular case to explore is that of cylindrical irregularities of various heights h and radii R:

$$f(r)=\begin{cases} h & \text{if } r\leqslant R\\ 0 & \text{if } r>R.\end{cases}$$

At all points, except for $r=R$, equations (4.10), (4.11) become

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}B}{\mathrm{d}r}\right)-\frac{1}{r^2}B=0,\qquad \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}C}{\mathrm{d}r}\right)-\frac{1}{r^2}C=0$$

and can be immediately solved. Taking into account the boundary conditions at $r=0,\infty$, we obtain

$$B(r)=\begin{cases} B_-r & \text{if } r<R\\ B_+r^{-1} & \text{if } r>R,\end{cases}\qquad C(r)=\begin{cases} C_-r & \text{if } r<R\\ C_+r^{-1} & \text{if } r>R,\end{cases}$$

where $B_\pm$ and $C_\pm$ are constants. We shall require B and C to be continuous at $r=R$:

$$B(R+0)=B(R-0),\qquad C(R+0)=C(R-0),$$

which implies

$$B_-R=B_+R^{-1},\qquad C_-R=C_+R^{-1}. \tag{5.1}$$

In order to derive the matching condition for $\mathrm{d}B/\mathrm{d}r$ and $\mathrm{d}C/\mathrm{d}r$, we observe that

$$\frac{\mathrm{d}f}{\mathrm{d}r}=-h\delta(r-R),$$

where $\delta(r)$ is the Dirac delta-function. Next, integrating (4.10), (4.11) over the infinitesimal interval $(R-0,R+0)$ we obtain

$$\omega\left[\left.\frac{\mathrm{d}B}{\mathrm{d}r}\right|_{r=R+0}-\left.\frac{\mathrm{d}B}{\mathrm{d}r}\right|_{r=R-0}\right]-\frac{\mathrm{i}}{R}hC(R)=-kh,$$

$$\omega\left[\left.\frac{\mathrm{d}C}{\mathrm{d}r}\right|_{r=R+0}-\left.\frac{\mathrm{d}C}{\mathrm{d}r}\right|_{r=R-0}\right]+\frac{\mathrm{i}}{R}hB(R)=lh,$$

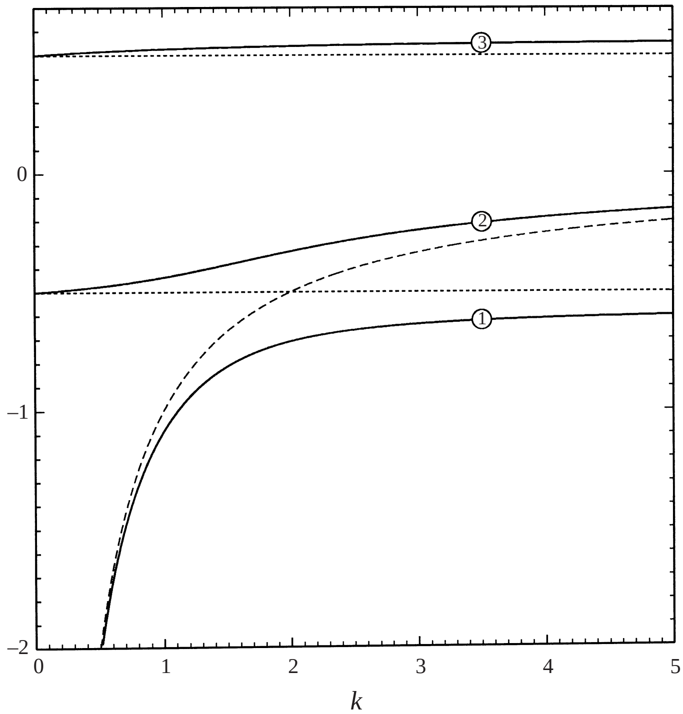

FIGURE 2. The dispersion curves for the case of cylindrical irregularities of identical heights (dispersion relation (5.6) with $h_0 = 1$, $A = 0.15$, $\alpha = 1$, $l = 0$). Curves 1 and 2 show mixed topographic–Rossby modes, curve 3 shows a purely topographic mode. The dashed line shows the unperturbed (flat-bottom) dispersion curve of the barotropic Rossby waves, the dotted lines show the 'topographic' frequency $\omega_{topo} = \pm\frac{1}{2}h$, where h is the height of irregularities.

which yields

$$\omega(-B_+R^{-2} - B_-) - \mathrm{i}hC_- = -kh, \qquad \omega(-C_+R^{-2} - C_-) - \mathrm{i}hB_- = lh. \tag{5.2}$$

Solving (5.1) and (5.2) for $B_\pm$, $C_\pm$, we have

$$B_+ = \frac{(-2\omega k - \mathrm{i}lh)h}{h^2 - 4\omega^2}, \qquad B_- = \frac{(-2\omega k - \mathrm{i}lh)hR^2}{h^2 - 4\omega^2}, \tag{5.3}$$

$$C_+ = \frac{(2\omega l - \mathrm{i}kh)h}{h^2 - 4\omega^2}, \qquad C_- = \frac{(2\omega l - \mathrm{i}kh)hR^2}{h^2 - 4\omega^2}. \tag{5.4}$$

Now, we substitute $B(r)$, $C(r)$, and $f(r)$ into (4.9) and obtain

$$2\pi\rho\omega\overline{\frac{R^2h^2}{h^2 - 4\omega^2}} + \frac{\alpha k}{k^2 + l^2} + \omega = 0 \tag{5.5}$$

(recall that the bar denotes averaging over the ensemble of allowable R and h).

5.1. *Periodic topography*

In this case all irregularities have the same height and radius, and the bar in equation (5.5) can be omitted:

$$\frac{2Ah^2\omega}{h^2 - 4\omega^2} + \frac{\alpha k}{k^2 + l^2} + \omega = 0, \tag{5.6}$$

where

$$A = \pi R^2 \rho$$

represents the proportion of the area of the bottom covered with irregularities. Observe that no dispersion curve $\omega(k)$ can cross the lines $\omega = \pm\frac{1}{2}h$ (if it could, the first term in (5.6) would be infinite with no other term to balance it). This conclusion is illustrated by figure 2, where (5.6) was solved numerically. The upper curve is entirely associated with topography. The middle curve is associated with topography only in the small-k region, and approaches the Rossby-wave curve for large k. The lower curve exhibits the opposite behaviour.

5.2. *Random topography*

In this case, it is convenient to introduce the distribution $\Gamma(R,h)$ of heights and radii of cylindrical irregularities and rewrite (5.5) as

$$2\pi\rho\omega \int_0^\infty\int_{-\infty}^\infty \Gamma(R,h)\frac{R^2h^2}{h^2-4\omega^2}\mathrm{d}h\,\mathrm{d}R + \frac{\alpha k}{k^2+l^2} + \omega = 0. \tag{5.7}$$

The simplest particular case to consider is

$$\Gamma(R,h) = \begin{cases} \dfrac{1}{2h_0R_0} & \text{if} \quad -h_0 < h < h_0, \quad 0 < R < R_0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.8}$$

Substituting (5.8) into (5.7) and evaluating the integrals with respect to h, we obtain

$$2A\omega\left(1 + \frac{2\omega}{h_0}\ln\frac{2\omega-h_0}{2\omega+h_0}\right) + \frac{\alpha k}{k^2+l^2} + \omega = 0, \tag{5.9}$$

where $A = \rho\pi\overline{R^2} = \frac{1}{3}\pi\rho R^2$ is, as before, the proportion of the area of the bottom covered with irregularities. The numerical solution to equation (5.9) is shown in figure 3. One can see that no curves are located within the band $-\frac{1}{2}h_0 \leqslant \omega \leqslant \frac{1}{2}h_0$.

6. Discussion

First, note that

$$\omega_{topo} = \tfrac{1}{2}h$$

is the frequency of the first topographic eigenmode supported by an isolated cylindrical irregularity (Jansons & Johnson 1988). Thus, the non-existence of free waves with $\omega = \omega_{topo}$ can be explained since they get captured by the irregularities as if those were 'resonators' tuned to the frequency of the incident wave. This argument applies equally to the periodic and random cases.

In the periodic case, our asymptotic method implies a restriction: ω should not be close to ω_{topo}. This follows from the fact that the eigenfunctions B and C become singular as $\omega \to \omega_{topo}$ (see (5.3)–(5.4)). Thus, those parts of the dispersion curves in figure 2 that are close to ω_{topo} are not reliable. Still, one can be sure that the exact curves remain close to ω_{topo} – because if they did not, our method would have been valid for them and shown their location elsewhere.

Rhines & Bretherton (1973) demonstrated that, for the case of one-dimensional random topography, the asymptotic method they (and everybody else after them) used fails. Specifically, the short-wave component of the wave field included a divergent

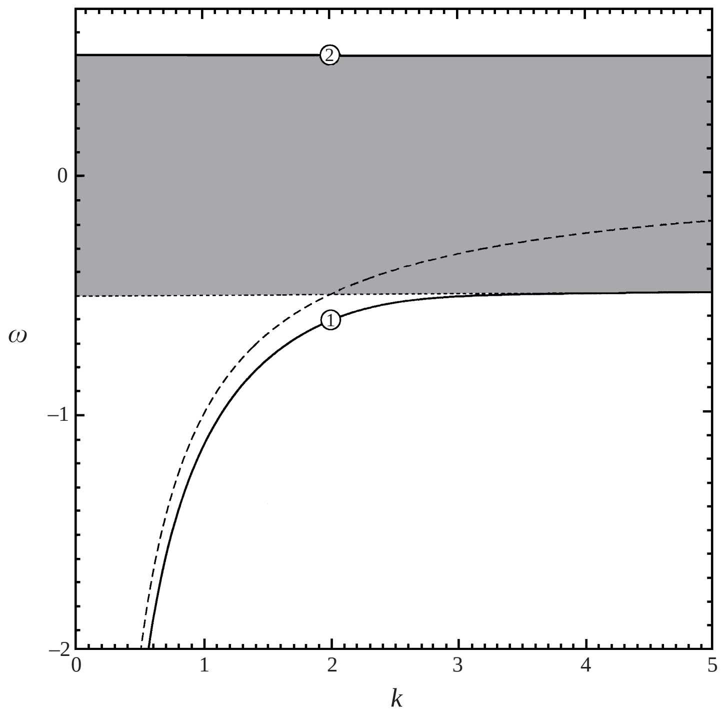

FIGURE 3. The dispersion curves for the case of irregularities with randomly distributed (within an interval $-h_0 < h < h_0$) heights (dispersion relation (5.8) with $h_0 = 1$, $A = 0.15$, $\alpha = 1$, $l = 0$). Curve 1 shows the topographic–Rossby mode, curve 2 shows the purely topographic mode. The dashed line shows the unperturbed (flat-bottom) dispersion curve of the barotropic Rossby waves. The area 'filled' with topographic frequencies is shaded.

integral:

$$\overline{\left(\phi^{(1)}\right)^2} = \infty.$$

In the present case, however, $\phi^{(1)}$ is finite. Indeed, using (4.2), one can show that

$$\overline{\left(\phi^{(1)}\right)^2} \approx \overline{\left[\sum_{n=1}^{\infty} \phi_n^{(1)}(r_n)\right]^2} \approx \overline{\sum_{n=1}^{\infty} \left[\phi_n^{(1)}(r_n)\right]^2} \qquad (6.1)$$

(recall that r_n is the distance between the nth irregularity and the current point). It can be shown that

$$\phi_n^{(1)} = O(r_n^{-1}) \quad \text{as } r \to \infty.$$

Accordingly, if the irregularities are distributed spatially homogeneously, (6.1) converges.

It is interesting to estimate the dimensional value corresponding to the characteristic topographic frequency

$$\omega_{topo} = \tfrac{1}{2}h. \qquad (6.2)$$

Assuming that the depth of the ocean is 5000 m, the latitude is 30°, and the height of the topography is 200 m, we obtain the period of 50 days.

Thus, our results suggest that some long-period oceanic oscillations do not propagate freely and are trapped by topographic features. It should be admitted, however, that we are at present unable to predict the proportion of trapped waves in the balance of oceanic wave energy, as this would require a much more realistic calculation than the one presented.

Finally, we shall discuss what happens to a disturbance if the distribution of the heights of irregularities covers all values, $-\infty < h < \infty$. In this case, the 'prohibited' band expands to infinity and no harmonic waves appear to be possible.

Indeed, observe that, for real ω, the integral in the first term of the dispersion relation (5.7) diverges at $h = \pm 2\omega$ – which is why (5.7) may not have real solutions. Nor can (5.7) have solutions with positive imaginary part ($\mathrm{Re}\,\omega > 0$), as this would correspond to a growing (unstable) wave and thus be inconsistent with energy conservation. Finally, (5.7) does not have solutions with negative imaginary part – for, if it does, one can readily show that the conjugate ω^* also satisfies (5.7), which would be inconsistent with energy conservation.

In order to find out how the wave evolves in this case, one needs to examine the initial-value problem for the wave field (instead of assuming the harmonic behaviour from the start). This is a much more difficult task than what we have done in this paper; however, one can speculate on it by using the analogy with the well-known examples of disturbances in plane Couette flow (Case 1960) and the Landau damping of waves in plasma (e.g. Landau & Lifshitz 1979). Similarly to the problem at hand, no harmonic modes (discrete spectrum) exist in those cases, and the evolution of an initial disturbance is described by a continuous spectrum. Still, the solution can be represented approximately by a harmonic wave, with a complex frequency corresponding to slow exponential decay. Using the analogy with the Couette flow (Case 1960) and the analytical properties of the left-hand side of (5.5), one can obtain an estimate of this frequency:

$$\omega \approx \omega_{Ro} - 2\mathrm{i}\pi A\omega_{Ro}^2 \left[\int_{-\infty}^{\infty} \Gamma(R, 2\omega_{Ro})\,\mathrm{d}R + \int_{-\infty}^{\infty} \Gamma(R, -2\omega_{Ro})\,\mathrm{d}R \right], \tag{6.3}$$

where $\Gamma(R,h)$ is the distribution of R and h, and

$$\omega_{Ro} = -\frac{\alpha k}{k^2 + l^2}$$

is the frequency of Rossby waves over a flat bottom. As expected (6.3) corresponds to a slowly decaying Rossby-wave oscillation. Mathematically, this decay is due to interlacing frequencies of the modes of a continuous spectrum; and from a physical viewpoint, the coherent harmonic wave loses energy to random oscillations captured by isolated topographic features.

This question deserves more rigorous investigation.

7. Conclusions

We have examined the linear waves on the beta-plane over sparse topography, i.e. over sparsely distributed, radially symmetric bottom irregularities. A dispersion relation for the frequency of the eigenmodes was derived. As a particular example of radially symmetric irregularities, we considered randomly/periodically distributed cylinders of various heights and radii.

The main results of the paper are as follows (the dimensional variables are hereinafter marked with hats).

(i) If cylindrical irregularities are of the same height $\widehat{h}$, no eigenmode can have its frequency equal to the 'topographic' frequency,

$$\widehat{\omega}_{topo} = \pm\frac{f\widehat{h}}{2H_0},$$

where H_0 is the mean depth of the ocean and f is the Coriolis parameter (this equality is the dimensional equivalent of (6.2)).

(ii) Physically, $\widehat{\omega}_{topo}$ is the 'natural frequency' of an oscillation trapped by an individual irregularity, which explains why free waves cannot have this frequency (they get captured by topography).

(iii) If the height of cylindrical irregularities is randomly distributed between $-\widehat{h}_0$ and $\widehat{h}_0$, the frequency of the eigenmodes cannot be within the band

$$\left(-\frac{f\widehat{h}_0}{2H_0}, \frac{f\widehat{h}_0}{2H_0}\right)$$

(for the same reason as in the case of irregularities of identical height).

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